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Order preserving assignments without contiguity

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Abstract

A variation of the order preserving assignment problem introduced in [8] is studied. The complete linear description of the associated polytope is derived and a polynomial-time separation algorithm for the describing inequalities is presented. A polynomial-time direct combinatorial algorithm based on dynamic programming and a longest path formulation of the problem are also given.

The order preserving assignment problem introduced in [8] is defined as follows: given a list of n ordered items, p positions and a profit c_{ij} for assigning item i to position j, find a profit maximizing assignment that preserves the order of the items and ensures the contiguity of the positions occupied. In [8] an ideal description of the polytope associated with this problem and a linear-time algorithm for the problem are given.

Here we examine the problem that results when the contiguity condition for the assignment is dropped. Besides being interesting on its own, this problem arises, for instance, in VLSI design, more precisely in the so-called programmable logic arrays (PLA) folding problem; see e.g. [3]. The way the problem occurs, is in the form of a subproblem of a large integer linear programming problem having a substantial set of complicating linear constraints that are not of the assignment type. In particular, using Lagrangian relaxation one needs to solve the 'pure' order preserving assignment problem without contiguity as a subproblem many times to get 'good' upper bounds to the larger VLSI problem. To develop a branch-and-cut algorithm to solve the large problem *directly*, the knowledge of an ideal, i.e. minimal and complete, description of the polytope associated with the subproblem along with separation algorithms for constraint identification are necessary. This is what we discuss in this paper.

In the next section we give some definitions and in Section 2 we give the ideal description of the associated polytope and a polynomial-time separation algorithm for the nontrivial facet defining inequalities. Finally, in Section 3 we present a polynomial-time dynamic programming algorithm to solve this problem and we formulate the problem as a longest path problem on a directed network.

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1. Some definitions

The problem is defined on a complete bipartite graph $G = (N,P,E)$, where $N = \{1,\dots,n\}$, $P = \{1,\dots,p\}$ and $1 \leqslant p \leqslant n$. An edge of G is an ordered pair $(k,t) \in N \times P$ and is denoted by a single letter ($e, a, \dots$ etc.) or by its defining pair of indices.

Definition 1. $A \subseteq E$ is a noncontiguous order preserving assignment (N-OPA) if

(i) for every $k \in N$ there is at most one $a \in A$ such that $k \in a$.

(ii) for every $t \in P$ there is at most one $a \in A$ such that $t \in a$.

(iii) if $a = (k,t), b = (\kappa,\tau) \in A$ and $a \neq b$, then either $k < \kappa$ and $t < \tau$ or $k > \kappa$ and $t > \tau$.

Properties (i) and (ii) ensure that A is an assignment, whereas property (iii) ensures that the assignment preserves the order of the items. We index the edges (k,t) of G sequentially when necessary by

$$e = (k-1)p + t, \tag{1}$$

where $1 \leqslant k \leqslant n$ and $1 \leqslant t \leqslant p$.

If $A \subseteq E$ is an N-OPA and $e = (k,t) \in A$, then we say that item k is assigned to position t. Let $\mathbb{R}^E$ (rather than $\mathbb{R}^{|E|}$) denote the space of real vectors of length $|E|$. The support of $x \in \mathbb{R}^E$ is the index set of the nonzero components of x. For $A \subseteq E$, we denote $x^A = (x^A_e) \in \mathbb{R}^E$ the characteristic vector (or incidence vector), i.e.,

$$x^A_e = \begin{cases} 1 & \text{if } e \in A, \\ 0 & \text{if } e \notin A. \end{cases}$$

We define the noncontiguous order preserving assignment polytope OP^{N}_{np} to be the convex hull of the characteristic vectors of all N-OPAs, i.e.

$$OP^{\mathrm{N}}_{np} = conv\{x^A \in \mathbb{R}^E \mid A \subseteq E \text{ is an N-OPA}\}.$$

The order preserving assignment problem then is the optimization problem

$$(OP^{\mathrm{N}}) \qquad \max\{c^{\mathrm{T}}x \mid x \in OP^{\mathrm{N}}_{np}\}.$$

It is trivial to show that OP^{N}_{np} is an *independence system*, i.e. if $x \in OP^{\mathrm{N}}_{np}$ with $x_{kt} = 1$ then x' defined by $x'_{kt} = 0$ and $x'_{ij} = x_{ij}$ for all other i, j satisfies $x' \in OP^{\mathrm{N}}_{np}$. However, $(E, \mathcal{N})$, where $\mathcal{N} = \{A \subseteq E : A \text{ is an N-OPA}\}$ is not a matroid as one can easily show (e.g. consider the elements $S = \{(1,1),(3,3)\}$ and $T = \{(1,1),(3,2),(4,3)\}$ of $\mathcal{N}$).

We say that two edges (i,j) and (k,t) of the graph G *cross* if either $i \leqslant k$ and $j \geqslant t$ or $i \geqslant k$ and $j \leqslant t$. A linear description of a polytope is a set of linear inequalities and/or equations whose set of solutions equals the polytope. A linear description is minimal, if none of the inequalities and/or equations can be dropped from it without changing the solution set, i.e. every inequality defines a facet of the polytope. We refer the reader to [7] for unexplained polyhedral terminology.

2. The facial structure of the polytope

Proposition 2. *The polytope OP^{N}_{np} is full dimensional, i.e. $\dim OP^{\mathrm{N}}_{np} = np$.*

Proof. Consider the np vectors $x^i \in OP^{\mathrm{N}}_{np}$ given by $x^i_e = 1$ if $e = i$ and $x^i_e = 0$ otherwise for all $e \in E$ and $1 \leqslant i \leqslant np$. The matrix which has these vectors as rows is a diagonal matrix and thus the vectors are linearly independent. Since $0 \in OP^{\mathrm{N}}_{np}$ we have $np + 1$ affinely independent vectors in OP^{N}_{np} and thus $\dim OP^{\mathrm{N}}_{np} = np$. □

We show next that all nonnegativity constraints

$$x_{ij} \geqslant 0 \quad \text{for all } (i,j) \in E \tag{2}$$

define facets — the 'trivial' facets — of OP^{N}_{np}.

Proposition 3. *All inequalities* (2) *define facets of OP^{N}_{np}.*

Proof. Consider the inequality $x_{kt} \geqslant 0$ for a given edge $(k,t) \in E$ and let $F = \{x \in OP^{\mathrm{N}}_{np} : x_{kt} = 0\}$. The inequality is clearly valid. All the vectors used in the proof of Proposition 2 except the vector $x^{(k-1)p+t}$ are in F. Therefore, F is a proper face of OP^{N}_{np} and there are np affinely independent vectors in F. Thus $\dim F = np - 1$, i.e. the inequality $x_{kt} \geqslant 0$ defines a facet of OP^{N}_{np}. □

To characterize the nontrivial facets of OP^{N}_{np} we consider the set $\mathscr{S}$ of all distinct subsets $S \subseteq E$ such that any two edges in S cross, i.e.

$$\mathscr{S} = \{S \subseteq E : (i,j) \neq (k,t) \in S \Rightarrow \text{either } i \leqslant k \text{ and } j \geqslant t \text{ or } i \geqslant k \text{ and } j \leqslant t\}.$$

An element $S \in \mathscr{S}$ is called *maximal* if $|S| = \max_{T \in \mathscr{S}} |T|$. The next lemma establishes some fundamental properties about the set $\mathscr{S}$.

Lemma 4. (i) *Let $S \in \mathscr{S}$ be a maximal element. Then the edges $(i_k, j_k) \in S$ can be indexed sequentially as follows: if $i_k = i_{k+1}$ then $j_{k+1} = j_k - 1$ and if $i_k < i_{k+1}$ then $i_k = i_{k+1} - 1$ and $j_k = j_{k+1}$.*

(ii) *The maximal elements S of $\mathscr{S}$ have cardinality $n + p - 1$ and $S \in \mathscr{S}$, $|S| = n + p - 1$ imply $(1,p) \in S$ and $(n,1) \in S$.*

(iii) *There are exactly $\binom{n+p-2}{p-1}$ distinct maximal elements in $\mathscr{S}$.*

(iv) *For every set S of edges ordered as in* (i), *with $(1,p)$ as the first edge and $(n,1)$ as the last one, we have that $S \in \mathscr{S}$, i.e. all edges in S cross.*

(v) *For every $K \in \mathscr{S}$ there exists a maximal set $S \in \mathscr{S}$ such that $K \subseteq S$.*

Proof. (i) Let $S \in \mathscr{S}$ and (i_k, j_k), $k = 1, \dots, K$ be the edges in S. Without restriction of generality we assume that $i_k \leqslant i_{k+1}$ and if $i_k = i_{k+1}$ that $j_{k+1} < j_k$ for $k = 1, \dots, K-1$. Now let (i_k, j_k), (i_{k+1}, j_{k+1}) be two edges of a maximal element $S \in \mathscr{S}$ with $i_k = i_{k+1}$ and suppose that $(i_k, j_k - 1) \notin S$. Since all edges in S cross we have that for any edge $(i_s, j_s) \in S$ other than (i_k, j_k) and (i_{k+1}, j_{k+1}) either $i_k \leqslant i_s$ and $j_s \leqslant j_{k+1} < j_k - 1$ or $i_s < i_k$ and $j_s \geqslant j_k > j_k - 1$, i.e. (i_s, j_s) crosses $(i_k, j_k - 1)$. So $(i_k, j_k - 1)$ crosses all edges in S which contradicts the assumed maximality of S and thus the first part of (i) follows. On the other hand, suppose that (i_k, j_k) and (i_{k+1}, j_{k+1}) are two edges in a maximal set $S \in \mathscr{S}$ with $i_k < i_{k+1}$. Since the two edges cross we have that $j_{k+1} \leqslant j_k$. If $i_k \neq i_{k+1} - 1$ or $j_k \neq j_{k+1}$ then $(i_k + 1, j_k) \notin S$. Suppose that this is the case and let (i_s, j_s) be any edge in S other than (i_k, j_k) and (i_{k+1}, j_{k+1}). Since (i_s, j_s) crosses both (i_k, j_k) and (i_{k+1}, j_{k+1}) we have that either $i_s \leqslant i_k < i_k + 1$ and $j_s \geqslant j_k$ or $i_s \geqslant i_{k+1} \geqslant i_k + 1$ and $j_s \leqslant j_{k+1} \leqslant j_k$, i.e. (i_s, j_s) crosses $(i_k + 1, j_k)$. It follows that $(i_k + 1, j_k)$ crosses all edges in S which again contradicts the maximality of S and thus part (i) follows.

(ii) From the ordering of the edges of S as in part (i) it follows that each $(i_k, j_k) \in S$, $k \geqslant 2$, is obtained from its predecessor (i_{k-1}, j_{k-1}) by either increasing i_{k-1} by one or by decreasing j_{k-1} by one. If we denote by α the number of times we increase the first endpoint, by β the number of times we decrease the second endpoint and by i_K and j_K the largest first endpoint and smallest second endpoint, respectively, we get that $i_K = i_1 + \alpha \leqslant n$ and $j_K = j_1 - \beta \geqslant 1$ and $|S| = \alpha + \beta + 1$. It follows that $\alpha \leqslant n - 1$, $\beta \leqslant p - 1$ and $\alpha + \beta \leqslant n + p - 2$, i.e. $|S| \leqslant n + p - 1$. The set $S = \{(1,p), (1,p-1), \dots, (1,1), (2,1), \dots, (n,1)\}$ shows that the upper bound of $n + p - 1$ is indeed attained. To complete the proof we note that for a maximal set we have $\alpha + \beta = n + p - 2$ and thus $i_1 = 1$ and $j_1 = p$, i.e. the edge $(1,p)$ is the first edge in every maximal set. Similarly for every maximal set $i_K = n$ and $j_K = 1$ and thus the edge $(n,1)$ is the last edge in every maximal set and the proof is complete.

(iii) From the proof of (ii) we have that $(1,p)$ must be the first edge and $(n,1)$ must be the last edge in S if the edges are ordered as in part (i). Starting with edge $(1,p)$ we perform $\alpha + \beta = n + p - 2$ consecutive changes of the endpoints of the edges to get to the edge $(n,1)$, $n-1$ of which are changes in the first endpoint and $p-1$ are changes in the second endpoint. Thus the number of distinct sets in $\mathscr{S}$ with $|S| = n + p - 1$ is equal to the number of permutations of $n + p - 2$ changes, $n - 1$ of the first endpoint and $p - 1$ of the second, i.e. is equal to $[(n+p-2)!]/[(n-1)!(p-1)!] = \binom{n+p-2}{p-1}$.

(iv) We prove the assertion by proving the claim that any edge (i_k, j_k) in an ordering as in the proof of (i) crosses all its predecessors for $k = 2, \dots, n + p - 1$. We prove the claim by induction. For $k = 2$ the claim is true since the second edge can be either $(1, p-1)$ or $(2, p)$ and in both cases it crosses $(1,p)$. Suppose the claim to be true for $k = m \geqslant 2$, i.e. all edges $(i_1, j_1), \dots, (i_m, j_m)$ cross. We have to prove that (i_{m+1}, j_{m+1}) crosses all edges (i_s, j_s) for $s = 1, \dots, m$. For every (i_s, j_s), $s = 1, \dots, m-1$, since (i_m, j_m) crosses all its predecessors we have either $i_s < i_m$ and $j_s \geqslant j_m$ or $i_s = i_m$ and $j_s > j_m$. Now if $i_{m+1} = i_m$ then $j_{m+1} = j_m - 1$ and thus either $i_s < i_{m+1}$ and $j_s \geqslant j_m > j_m - 1 = j_{m+1}$ or $i_s = i_{m+1}$ and $j_s > j_m > j_{m+1}$, i.e. (i_{m+1}, j_{m+1}) crosses (i_s, j_s). On the other hand, if $i_{m+1} = i_m + 1$ then $j_{m+1} = j_m$ and thus $i_s < i_{m+1}$ and $j_s \geqslant j_{m+1}$ or

$i_s < i_{m+1}$ and $j_s > j_{m+1}$, i.e. (i_{m+1}, j_{m+1}) crosses (i_s, j_s). So (i_{m+1}, j_{m+1}) crosses (i_s, j_s) for $s = 1, \dots, m-1$ and since it crosses (i_m, j_m) too, the claim follows.

(v) If K is maximal then $S = K$ proves the assertion. So suppose that K is a set of edges in E such that all edges in K cross and let $(i_1, j_1), \dots, (i_k, j_k)$ be the elements of K, where $k = |K|$. Without restriction of generality we can assume that $i_s \leqslant i_{s+1}$ and if $i_s = i_{s+1}$ then $j_s > j_{s+1}$ for all $s = 1, \dots, k-1$. Define the following sets: $S_1 = \{(1,p), \dots, (1, j_1)\}$ if $j_1 < p$, $S_1 = \{(1,p)\}$ otherwise, $S_2 = \{(2, j_1), \dots, (i_1, j_1)\}$ if $i_1 > 2$, $S_2 = \{(2, j_1)\}$ if $i_1 = 2$, $S_2 = \emptyset$ otherwise, $S_\ell^1 = \{(i_\ell + 1, j_\ell), \dots, (i_{\ell+1}, j_\ell)\}$ if $i_\ell < i_{\ell+1}$, $S_\ell^1 = \emptyset$ otherwise, $S_\ell^2 = \{(i_{\ell+1}, j_\ell - 1), \dots, (i_{\ell+1}, j_{\ell+1})\}$ if $j_{\ell+1} < j_\ell$, $S_\ell^2 = \emptyset$ otherwise, for $\ell = 1, \dots, k-1$, $S_3 = \{(i_k, j_k - 1), \dots, (i_k, 1)\}$ if $j_k > 1$, $S_3 = \emptyset$ otherwise, $S_4 = \{(i_k + 1, 1), \dots, (n, 1)\}$ if $i_k < n$ and $j_k = 2$, $S_4 = \{(i_k, 1), \dots, (n, 1)\}$ if $i_k < n$ and $j_k > 2$ and $S_4 = \emptyset$ otherwise. Now the set $S = S_1 + S_2 + \sum_{\ell=1}^{k-1} (S_\ell^1 + S_\ell^2) + S_3 + S_4$ has cardinality of $n + p - 1$ and by construction all of its elements are ordered as in (i). Thus, from part (iv) we have $S \in \mathscr{S}$ and the lemma follows. □

Consider now the following inequality (3) which is clearly valid for OP_{np}^{N}

$$\sum_{(i,j) \in S} x_{ij} \leqslant 1 \quad \text{for } S \in \mathscr{S} \text{ and } |S| = n + p - 1. \tag{3}$$

The next proposition shows that it is also facet defining and since there is one inequality for every maximal set $S \in \mathscr{S}$ from part (iii) of Lemma 4 we get that there are precisely $\binom{n+p-2}{p-1}$ distinct facets of the form (3).

Proposition 5. *Every inequality* (3) *defines a facet of* OP_{np}^{N}.

Proof. Let $S \in \mathscr{S}$ be such that $|S| = n + p - 1$ and $F = \{x \in OP_{np}^{\mathrm{N}} : \sum_{(i,j) \in S} x_{ij} = 1\}$. Clearly F is a nonempty, proper face of OP_{np}^{N}. Suppose that it is not a facet of OP_{np}^{N}. Then there exists an inequality $bx \leqslant b_0$ such that $F \subset F_b = \{x \in OP_{np}^{\mathrm{N}} : bx = b_0\}$ and F_b is a facet of OP_{np}^{N}. Consider any two edges (k,t), $(\ell, s) \in S$ and the vectors x with $x_{\ell s} = 1$ and $x_{ij} = 0$ otherwise and x' with $x'_{kt} = 1$ and $x'_{ij} = 0$ otherwise which are both in F and thus in F_b. It follows that $b_{kt} = b_{\ell s} = b_0 \neq 0$ for any two edges (k,t), $(\ell, s) \in S$. Therefore, $F_b = \{x \in OP_{np}^{\mathrm{N}} : \sum_{(i,j) \notin S} b_{ij} x_{ij} + \sum_{(i,j) \in S} x_{ij} = 1\}$. For any edge $(m,r) \notin S$ there exists an edge $(k,t) \in S$ such that the assignment $x_{mr} = x_{kt} = 1$ and $x_{ij} = 0$ otherwise is feasible. For, suppose not. Then (m,r) crosses all edges in S and thus $S \cup (m,r) \in \mathscr{S}$ which contradicts the maximality of S. So, consider the vectors x with $x_{mr} = x_{kt} = 1$ and $x_{ij} = 0$ otherwise and x' with $x'_{kt} = 1$ and $x'_{ij} = 0$ otherwise. We have $x, x' \in F \subset F_b$ and thus we get $b_{mr} = 0$ for all $(m,r) \notin S$. It follows that $F = F_b$ which contradicts the assumption that F is not a facet, and the proposition follows. □

The following proposition shows that inequalities (2) and (3) furnish a zero-one formulation of OP_{np}^{N}.

Proposition 6. *Let $Q = \{x \in \{0,1\}^E:\ x$ satisfies (2) and (3)$\}$. Every zero-one vector of Q is an* N-OPA *and vice versa, i.e. every N-OPA corresponds to a zero-one point in Q.*

Proof. From the validity of (2) and (3) it follows that $x \in Q$ for every incidence vector x of an N-OPA. Suppose now that $x \in Q$. We show that x is the incidence vector of an N-OPA by showing that it satisfies the three conditions of Definition 1. Consider any $k \in N$ and let $C_k = \{(k,i) \in E:\ i \in P\}$. Since all edges in C_k cross we have that $C_k \in \mathscr{S}$ and thus $\sum_{i \in P} x_{ki} \leqslant 1$ by part (v) of Lemma 4 since $x \in Q$. So condition (i) is satisfied. Similarly, for condition (ii) consider any $t \in P$ and let $C_t = \{(i,t) \in E:\ i \in N\}$; we conclude that $\sum_{i \in N} x_{it} \leqslant 1$ for all $x \in Q$. So let $x \in Q$ and suppose that conditions (i) and (ii) hold but (iii) does not hold, i.e. suppose that $x_{kt} = x_{\kappa\tau} = 1$ for some k, $\kappa \in N$, t, $\tau \in P$ and that $k > \kappa$ and $t < \tau$ or $k < \kappa$ and $t > \tau$ or $k \leqslant \kappa$ and $t > \tau$ or $k > \kappa$ and $t \leqslant \tau$. In any case the edges (k,t) and (κ,τ) cross and thus by part (v) of Lemma 4 there exists $S \in \mathscr{S}$ such that (k,t), $(\kappa,\tau) \in S$ and $|S| = n + p - 1$. But then one of the inequalities (3) is not satisfied which contradicts the assumption that $x \in Q$. $\square$

From part (iii) of Lemma 4 using Stirling's formula we get the following estimate of the number of inequalities (3):

$$\left(\frac{n-1}{p-1}+1\right)^{p-1/2}\left(\frac{p-1}{n-1}+1\right)^{n-1/2}(2\pi(n+p-2))^{-\frac{1}{2}}.$$

Since $\binom{n+p-2}{p-1}$ is maximized when $n = p$ this number is bounded by $(2(n-1))^{\sqrt{n-1}}$ and is exponential in n and p. Naturally, it is not desired to have a large number of inequalities in the linear programming problem that has to be solved. Before addressing the question of completeness of the linear description of OP_{np}^{N} given so far we first present a separation algorithm for the identification of violated inequalities of the form (3).

Separation for inequalities (3). Consider the network $F = (M,A)$ with node set M consisting of $np + 1$ nodes each of which, except one which we call 'dummy' and denote by (D), corresponds to an edge of the graph G. An arc from node (i,s) to (k,t) is in A if either $i = k$ and $t = s - 1$ or $i = k - 1$ and $s = t$. There is also an arc going from node $(n,1)$ to the dummy node (D). Fig. 1 gives an illustration of the network F for the case $n = p = 3$. From part (i) of Lemma 4 it follows that every maximal set $S \in \mathscr{S}$ corresponds to a path from node $(1,p)$ to node $(n,1)$, and vice versa. That is, the sequence of nodes in the path is the sequence of the elements of the set S in the ordering of part (i) of Lemma 4.

To solve the problem using linear programming, one starts with any solution x with components between zero and one, and generates the inequalities (3) as needed. For example, we can start with the linear programming problem consisting of the

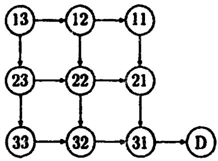

Fig. 1. The network F for $n = p = 3$.

assignment constraints $\sum_{i=1}^{n} x_{ij} \leqslant 1$ for $j = 1,\dots,p$ and $\sum_{j=1}^{p} x_{ij} \leqslant 1$ for $i = 1,\dots,n$ plus the nonnegativity constraints. Given the current linear programming solution x we let every arc in A that emanates from node (i,j) to have length x_{ij}. Then, the length of any path from node $(1,p)$ to node (D) is the value of the left-hand side of the inequality (3) for the corresponding set S. Therefore, by solving the longest path problem between nodes $(1,p)$ and (D) one finds a violated inequality of the form (3) if the length of the path is larger than 1 or concludes that the current solution is optimal otherwise. Since the network F is acyclic, the longest dipath problem is solvable in polynomial time, see [2], and thus we have the following proposition.

Proposition 7. *The separation problem for inequalities* (3) *is solvable in polynomial time.*

Let now $a = (i,s), b = (k,t) \in K \subseteq E$ and consider the binary relation $\ll$ defined as follows: $a \ll b \Leftrightarrow i \leqslant k$ and $s \geqslant t$.

Lemma 8. $(K, \ll)$ *is a* poset *(partially ordered set).*

Proof. It is trivial to verify that $\ll$ is reflexive (i.e. $a \ll a$), transitive (i.e. $a \ll b$ and $b \ll c \Rightarrow a \ll c$) and antisymmetric (i.e. $a \ll b$ and $b \ll a \Rightarrow a = b$) and thus $(K, \ll)$ is a poset. □

Note that all sets $T \in \mathscr{S}$ are totally ordered, i.e. all edges in T are pairwise comparable. These sets are called *chains*. Also, all sets T such that all edges in T do not cross are not ordered, i.e. all edges are pairwise incomparable. These sets are called *antichains*, see [9]. Lemma 8 can be extended to bipartite graphs with more than one edges between a pair of nodes (parallel edges). In this case, the reflexive and antisymmetric properties do not hold anymore. This difficulty is resolved by creating for each node as many copies as the maximum number of edges incident to it, ordering the 'new' nodes and 'filling-up' the edges from top to bottom contiguously, starting from the first 'new' node. In this way, we can guarantee that the relation $\ll$ remains reflexive and antisymmetric. We will use this fact in the proof of the next proposition which gives a characterization of the facets of OP_{np}^{N}. In particular, it shows that the nonnegativity inequalities are the only facet defining inequalities with right-hand side equal to zero and inequalities (3) the only other facets of OP^{N}_{np}.

Proposition 9. *Let $bx \leqslant b_0$ be a facet defining inequality of OP^{N}_{np} with b, b_0 integers. Then*

(i) $b_0 \geqslant 0$;

(ii) *if $b_0 = 0$ then $bx = -x_{kt}$ for some $(k,t) \in E$.*

(iii) *if $b_0 > 0$ then $bx \leqslant b_0$ is one of the inequalities* (3).

Proof. Part (i) follows immediately from the fact that $0 \in OP^{\mathrm{N}}_{np}$. To prove part (ii) suppose that $b_0 = 0$ and $bx \neq -x_{kt}$ for all $(k,t) \in E$. Since the inequality $bx \leqslant 0$ is valid and the solution x with $x_{kt} = 1$ for some $(k,t) \in E$ and $x_{ij} = 0$ otherwise is feasible, it follows that $b_{kt} \leqslant 0$ for all $(k,t) \in E$. But then, $bx \leqslant 0$ is a nonnegative combination of the nonnegativity inequalities and since by assumption it is not one of these inequalities it cannot be facet defining.

From parts (i) and (ii) we have that every facet defining inequality other than nonnegativity is of the form $bx = \sum_{e \in K} b_e x_e \leqslant b_0$ for some $K \subseteq E$, where $b_0 \neq 0$, b_e are integers. From the validity of such an inequality we have that $b_e \leqslant b_0$ and since OP^{N}_{np} is an independence system we have that $b_e \geqslant 0$ for all $e \in K$.

Let K' be obtained from K by considering b_e copies of each e of K. That is, the inequality $bx \leqslant b_0$ is written as $\sum_{e \in K'} x_e \leqslant b_0$ and thus by assumption $F_b = \{x \in OP^{\mathrm{N}}_{np}: \sum_{e \in K'} x_e = b_0\}$ is a facet of OP^{N}_{np}. By Lemma 8 and its extension to bipartite graphs with parallel edges (see above), $(K', \ll)$ is a poset. Since the inequality is facet defining the maximum cardinality of an antichain of $(K', \ll)$ is b_0 and by Dilworth's theorem (see e.g. [9]) the minimum number of disjoint chains that contain all edges in K' is b_0. It follows that the set K' can be partitioned into b_0 sets such that $K' = K_1 \cup \cdots \cup K_{b_0}$ and $K_i \in \mathscr{S}$ for $i = 1, \ldots, b_0$. But then by part (v) of Lemma 4 we have that there exist maximal sets $S_i \in \mathscr{S}$ such that $K_i \subseteq S_i$ for $i = 1, \ldots, b_0$. The inequality $\gamma x = \sum_{k=1}^{b_0} \sum_{(i,j) \in S_k} x_{ij} \leqslant b_0$ is valid for OP^{N}_{np} since it is the sum of b_0 inequalities of the form (3) and dominates the inequality $bx \leqslant b_0$ since by construction we have that $\gamma_{ij} \geqslant b_{ij}$ for all $(i,j) \in E$, which contradicts the assumption that F_b is a facet. So every inequality other than the nonnegativity is of the form (3) and the proposition follows. $\square$

As an immediate consequence of the previous proposition we have the following theorem.

Theorem 10. *Inequalities* (2) *and* (3) *furnish the ideal description of* OP^{N}_{np}.

From Theorem 10 it follows that the matrix that corresponds to inequalities (3) is a perfect zero-one matrix, see [5]. However, this matrix is not always balanced and thus, in particular, not totally unimodular in the general case, see [6]. The smallest matrix that shows this, is the one obtained for the case $n = p = 3$ and it is shown in Fig. 2.

$$\begin{pmatrix} 1&1&1&1&0&0&1&0&0 \\ 0&0&1&0&0&1&1&1&1 \\ 0&1&1&1&1&0&1&0&0 \\ 0&0&1&1&1&1&1&0&0 \\ 0&1&1&0&1&0&1&1&0 \\ 0&0&1&0&1&1&1&1&0 \end{pmatrix}$$

Fig. 2. The matrix of constraints (3) for $n = p = 3$.

Since the zero–one matrix that corresponds to inequalities (3) is perfect, its intersection graph, see [4], is a perfect graph. The intersection graph, H say, of the matrix can be constructed from the bipartite graph $G = (N, P, E)$ as follows. $H = (E, X)$ has as node set the edge set of G and has an edge connecting two nodes in E if the corresponding edges in G cross, i.e. X is the set of pairs of crossing (comparable) edges in E. The following proposition shows that the complement graph $\overline{H}$ of H is a *comparability graph*, i.e. its edges can be oriented to obtain a transitive acyclic digraph, and thus both H and $\overline{H}$ are perfect, see [2].

Proposition 11. *The graph $\overline{H}$ is a comparability graph.*

Proof. Let $\overline{H} = (E, Y)$ where Y is the set of pairs of noncrossing edges in E and $u = (u_1, u_2)$, $v = (v_1, v_2)$ be noncrossing edges, i.e. $(u, v) \in Y$. We orient the edge to be *from u to v* if $u_1 < v_1$. Now let $(u, v), (v, w) \in Y$. We prove that $(u, w) \in Y$ and thus with the given orientation the graph is transitive acyclic. Since u, v do not cross and $u_1 < v_1$ we have $u_2 < v_2$ and since v, w do not cross and $v_1 < w_1$ we have $v_2 < w_2$. It follows that $u_1 < w_1$ and $u_2 < w_2$. Hence, $(u, w) \in Y$ and the proposition follows. □

Inequalities (3) are the clique constraints and thus the linear description (2) and (3) is the description of the weighted stable set problem on a perfect graph, see [2,5]. Hence one could prove that (2) and (3) form an ideal description of OP^{N}_{np} by proving that H is perfect, using the above proposition. The perfectness of H can also be proven using Dilworth's theorem, see the proof of Proposition 9.

The noncontiguous order preserving assignment problem is the weighted stable set problem on the graph H where the weight of a node is the 'profit' of the corresponding edge in G. The weighted stable set problem can in principle be solved in polynomial time for perfect graphs, see [2], by a general algorithm for stable sets in perfect graphs that uses among other constructions the ellipsoid method. Thus in view of the exponentiality of the constraint set, a direct algorithm to this variant of our problem is needed. Although there exist specialized algorithms that solve the weighted stable set and maximum clique problems on special perfect graphs — comparability graphs included —, see e.g. [1], to use such an algorithm for our problem requires that we first construct the graph H from the given bipartite graph G. In the following section we present a polynomial time algorithm based on dynamic programming which works on the given graph G. Also, we give a reformulation of the noncontiguous order-preserving assignment problem as a longest path problem on a directed network.

3. A dynamic programming algorithm and a longest path formulation

The order preserving noncontinguous assignment problem can be solved by a dynamic programming algorithm the complexity of which is easily verified to be $\mathcal{O}(np)$, i.e. the algorithm is linear in the number of edges of G. Define $f(i,t)$ to be the maximum profit of assigning some of the first i items to positions $1,\ldots,t$, where $1 \leqslant i \leqslant n$, $1 \leqslant t \leqslant p$. We initialize $f(0,t) = 0$ for all t. The optimal assignment will have a profit of

$$\max\left\{0;\ \max_{\substack{1\leqslant t\leqslant p\\ 1\leqslant i\leqslant n}} f(i,t)\right\}$$

where the recursion formula is

$$f(i,t) = \max\{f(i-1,t); f(i,t-1); c(i,t) + f(i-1,t-1)\}.$$

In view of the above algorithm, the problem can also be formulated as a longest path problem on a directed network as follows. Define the network $H = (V,A)$ with the node set V and the arc set A defined as follows. Each edge $(i,j) \in E$ of the bipartite graph G of Section 1 gives rise to a node in V. There are two more nodes, the 'source' node denoted by s and the 'sink' node denoted by t. There is an arc from node s to every other node in V and an arc from every node in V except t to node t. Finally, there exists an arc from node (i,j) to node (k,t) with $i < k$ if the edges (i,j) and (k,t) do not cross. Assuming that all arcs emanating from node s have length of zero and that each arc emanating from node (i,j) has length c_{ij}, the noncontiguous order preserving assignment problem is the problem of finding the longest path from node s to node t.

This formulation allows us to solve the problem even when we have *interaction* costs, i.e. when we incur a cost c_{ij}^{kt} from making the assignment (i,j), (k,t). (The dynamic programming algorithm solves this problem, too.)

To solve this problem we have to redefine the length of each arc. The 'individual' cost from making the assignment (i,j) is taken into account by replacing the node (i,j) by two nodes (i_1,j_1) and (i_2,j_2) and introducing an arc from node (i_1,j_1) to node (i_2,j_2) with length c_{ij}. The length of the arc between nodes (i_2,j_2) and (k_1,t_1) is c_{ij}^{kt}.

Finally, we note that the problem solved in [8] can be formulated similarly as a longest path problem on the network H, with the (obvious) appropriate changes in the definition of the arc set A.

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